

Determinantal invariant gravity

Nurettin Pirinçcioğlu

February 16, 2016

Abstract

Einstein-Hilbert action with a determinantal invariant has been considered. The obtained field equation contains the **inverse Ricci tensor**, $\mathfrak{R}_{\alpha\beta}$. The linearized solution of invariant has been examined, and constant curvature space-time metric solution of the field equation gives different curvature constant for each values of σ . $\sigma = 0$ gives a trivial solution for constant curvature, R_0 .

1 Introduction

Observations of the universe accumulate many investigation on Einstein theory of general relativity. One of them is the modification of Einstein-Hilbert (EH) action. There are numerous investigations on the modified EH action with different context [1]. Such modifications cast a vital role in the inflationary cosmological model of Starobinsky [2]. Modeling the exponentially expanding the early universe, *i.e.* the inflation, is the most capable theory to explain the natural structure of the present universe; such as horizon, flatness, isotropy, homogeneity *etc.*. There are various inflationary models of the universe were introduced by different studies with different context [3, 4, 5, 6, 7]. For more information one can see the review [8] and references there in. According to the Planck observations [9], the most working model of inflation is that of Starobinsky. Our goal in this paper is to construct a determinantal invariant parameter which can be used in the EH action. The constant curvature solution of the invariant coincides with Starobinsky cosmological inflationary model. Also, constant curvature space-time solution of equation has been examined.

One can construct such a determinantal invariant with the same analogy in [10, 11]. The ratio of determinant of the Ricci tensor and metric tensor [12] is

$$r = \frac{\tilde{R}}{g} \quad (1)$$

Where \tilde{R} is the determinant of Ricci tensor, $R_{\mu\nu}$, g is that of metric tensor, $g_{\mu\nu}$. This parameter is our determinantal invariant. Accordingly a parameter function can be given as

$$f(r) = \xi \frac{(r)^\sigma}{M^{4(2\sigma-1)}} \quad (2)$$

Where σ , ξ are dimensionless constant numbers. σ is relating mass parameter, M , and determinantal parameter, \tilde{R} to fix f as a dimensionfull parameter. One can construct an action integral of EH with the determinantal invariant function, $f(r)$, as follows

$$S = \int d^4x \sqrt{-g} \left\{ \frac{M_{pl}^2}{2} R + \xi \frac{1}{M^{4(2\sigma-1)}} \left(\frac{\tilde{R}}{g} \right)^\sigma \right\} + S_{matter}. \quad (3)$$

Where R is curvature scalar. Variation of equation (3) with respect to the metric tensor produces the field equation

$$G_{\mu\nu} + \frac{1}{M_{pl}^2} g_{\mu\nu} (2\text{rd} - 1) f(r) + \frac{\sigma}{M_{pl}^2} \{ g_{\mu\nu} \nabla_\alpha \nabla_\beta [f \mathfrak{R}^{\alpha\beta}] + \nabla_\alpha \nabla^\alpha [f \mathfrak{R}_{\mu\nu}] - 2 \nabla_\mu \nabla_\alpha [f \mathfrak{R}_\nu^\alpha] \} = \frac{1}{M_{pl}^2} T_{\mu\nu}. \quad (4)$$

Where d is derivative with respect to the determinantal invariant, r , and $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ is the Einstein tensor. The equation (4) has the novel structure, because it contains *inverse Ricci* tensor, $\mathfrak{R}^{\alpha\beta}$. Comparing with the $f(R)$ gravity theories [1], our field equation is very different, because of it has three extra terms with *inverse Ricci* tensor, $\mathfrak{R}^{\alpha\beta}$. $\sigma = 0$ case, simplifies the field equation as follows

$$G_{\mu\nu} - \xi \frac{M^4}{M_{pl}^2} g_{\mu\nu} = \frac{1}{M_{pl}^2} T_{\mu\nu}. \quad (5)$$

The vacuum solution of this equation is the maximally symmetric solution of field equation (4).

2 Linearized solution

In the linearized approximation (up to the first order of $h_{\mu\nu}$) the metric tensor $g_{\mu\nu}$, and its inverse $g^{\mu\nu}$ become

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad (6)$$

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + \frac{1}{2} h^{\mu\alpha} h_\alpha^\nu. \quad (7)$$

Where $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ is the flat Minkowski space-time metric. In the Minkowski background the linearized Ricci tensor, and curvature scalar, become

$$R_{\mu\nu} = \frac{1}{2} (\partial_\alpha \partial_\mu h_\nu^\alpha + \partial_\alpha \partial_\nu h_\mu^\alpha - \partial_\mu \partial_\nu h - \partial^\alpha \partial_\alpha h) \quad (8)$$

$$R = g^{\mu\nu} R_{\mu\nu} = \partial_\mu \partial_\nu h^{\mu\nu} - \partial^\alpha \partial_\alpha h \quad (9)$$

respectively. In this section we consider the behavior of field equation (4) in the linearized approximation. The linearized form (expanding f up to the first order of $h_{\mu\nu}$) of determinantal invariant is

$$f = -\frac{\tilde{R}_{lin}}{M^4(1+h+\dots)} \approx -M^{-4}\tilde{R}_{lin} = f_{lin} \quad (10)$$

for $\sigma = 1$. Where \tilde{R}_{lin} is the determinant of the linearized Ricci tensor. Using empty space condition for energy momentum tensor of matter, $T_{\mu\nu} = 0$, the linearized solution [11, 13] of equation (4) in the Minkowski background, expanding determinantal potential about $r = 0$, is obtained as

$$G_{\mu\nu}^{lin} = \frac{1}{M_{pl}^2} t_{\mu\nu} \quad (11)$$

Where $G_{\mu\nu}^{lin}$ is the linearized Einstein tensor, and $t_{\mu\nu}$ is the 1st order perturbed (gravitational field) energy momentum tensor.

$$G_{\mu\nu}^{lin} = \frac{1}{2}(\partial_\alpha \partial_\nu h_\mu^\alpha + \partial_\alpha \partial_\mu h_\nu^\alpha - \partial_\mu \partial_\nu h - \partial^\alpha \partial_\alpha h_{\mu\nu} - \eta_{\mu\nu} \partial_\alpha \partial_\beta h^{\alpha\beta} + \eta_{\mu\nu} \partial^\alpha \partial_\alpha h) \quad (12)$$

The linearized Einstein field (11) equation takes the form of

$$\partial_\alpha \partial_\nu h_\mu^\alpha + \partial_\alpha \partial_\mu h_\nu^\alpha - \partial_\mu \partial_\nu h - \partial^\alpha \partial_\alpha h_{\mu\nu} - \eta_{\mu\nu} \partial_\alpha \partial_\beta h^{\alpha\beta} + \eta_{\mu\nu} \partial^\alpha \partial_\alpha h = \frac{2}{M_{pl}^2} t_{\mu\nu} \quad (13)$$

3 Constant curvature space-time solution

The space-time metric with constant curvature, R_0 , is characterized by the condition

$$R_{\mu\nu\alpha\beta} = \frac{1}{12} R_0 (g_{\mu\alpha} g_{\nu\beta} - g_{\mu\beta} g_{\nu\alpha}) \quad (14)$$

on the Riemann tensor. So, the Ricci tensor satisfies

$$R_{\mu\nu} = \frac{1}{4} R_0 g_{\mu\nu}. \quad (15)$$

from this, one can readily find the *inverse Ricci* tensor as follows

$$\mathfrak{R}^{\mu\nu} = \frac{4}{R_0} g^{\mu\nu}. \quad (16)$$

The maximally symmetric solution of equation (4) in vacuum is

$$\frac{1}{4}R_0g_{\mu\nu} - \frac{2}{M_{pl}^2}g_{\mu\nu}rf_r + \frac{1}{M_{pl}^2}g_{\mu\nu}f = 0. \quad (17)$$

Where

$$r = \frac{1}{256}R_0^4, \quad (18)$$

$$f(r) = \xi\left(\frac{R_0}{4M^2}\right)^{4\sigma}M^4, \quad (19)$$

and

$$df = f_r = \sigma\xi\left(\frac{R_0}{4M^2}\right)^{4(\sigma-1)}\frac{1}{M^4}. \quad (20)$$

Contracting the equation (17), gives us the algebraic

$$R_0 - \xi\frac{4}{M_{pl}^2}\left(\frac{R_0}{4M^2}\right)^{4\sigma}M^4(2\sigma - 1) = 0 \quad (21)$$

equation. The solution for R_0 is not trivial for all values of $f \neq 0$. But, one can get the trivial value of R_0 for $\sigma = 1/2$. $\sigma = 0$ gives us the coupling constant ξ which linearly related to the R_0 as follows

$$\xi = -\frac{M_{pl}^2}{4M^4}R_0. \quad (22)$$

This coupling constant is positive just for negative constant curvature, R_0 . Setting $\sigma = 1$, ξ becomes function of constant curvature, and Planck mass

$$\xi = M_{pl}^2M^4\left(\frac{4}{R_0}\right)^3. \quad (23)$$

This is positive just for positive values of R_0 .

In the case of $\sigma = 1/2$, the determinantal invariant, f , for 4-dimensional constant curvature space-time becomes

$$f_c = \xi\frac{1}{16}R_0^2. \quad (24)$$

This result is compatible with Starobinsky inflationary model [2], R^2 . Then the constant curvature solution of equation (4) is

$$-M_{pl}^2g_{\mu\nu}R_0 - \xi\frac{1}{8}g_{\mu\nu}R_0^2 = 0. \quad (25)$$

One can compare this result with the special case (constant curvature space-time) of Starobinsky inflationary parameter. The Starobinsky model of inflation can be written as follows

$$f_s(R) = R - \frac{1}{6m^2}R^2. \quad (26)$$

This is known as the chaotic inflationary model of Starobinsky, and it is perfectly well fitted with Planck data [9]. From the comparison of inflationary parameters of equation (25) with that of Starobinsky, the inflation mass can be given as

$$m \simeq M_{pl}/\sqrt{\xi}. \quad (27)$$

Inflaton mass [14] can be related to the reduced Planck mass with $\xi \sim 1$ limit in the early universe.

4 Conclusion

Determinantal invariant modification of EH action (1), does not affect the linearized solution, equation (13). However, constant curvature space-time solution of EH action with determinantal invariant, equation (24), mimics the Starobinsky inflationary parameter, R^2 , equation (26). The maximally symmetric solution of action (3) gives us very different results for coupling constant, ξ . Field equation (4) contains inverse Ricci tensor. Thus, the field equation (4) may produce novel results for physical or mathematical problems considered. As a result one can guess the mass of inflaton from equation (27).

References

- [1] A. Nunez and S. Solganik, Phys. Lett. B **608** (2005) 189. arXiv:0411120 [hep-th]; D. A. Easson, Int. J. Mod. Phys. A **19**, (2004) 5343. arXiv:0411209 [astro-ph]; T. Chiba, JCAP **03** (2005) 008. arXiv:0502070 [gr-qc]; Je-An Gu, Int. J. Mod. Phys. Conf. Ser. **10** (2012) 63.
- [2] A. A. Starobinsky, Phys. Lett. B **91** (1980) 99-102.
- [3] A. A. Starobinsky, JETP Lett. **30** (1979) 682 [Pisma Zh. Eksp. Teor. Fiz. **30** (1979) 719].
- [4] A. H. Guth, Phys. Rev. D **23** (1981) 347.
- [5] A. D. Linde, Rep. Prog. Phys. **47** (1984) 925.
- [6] A. Albrecht and P. J. Steinhardt, Phys. Rev. Lett. **48**, (1982) 1220.
- [7] F. Bezrukov, M. Shaposhnikov, Phys. Lett. B **659** (2008) 703. arXiv:0710.3755 [hep-th]. F. Bezrukov, arXiv:1307.0708 [hep-ph].
- [8] A. Linde, arXiv:0402051 [hep-th].
- [9] Planck Collaboration, arXiv:1303.5082 [astro-ph.CO]
- [10] D. A. Demir and N. K. Pak, Class. Quantum Grav. **26** (2009) 105018. arXiv:0904.0089 [hep-th].

- [11] N. Pirinccioglu, Gen. Relativ. Gravit. **44** (2012) 2563. arXiv:0908.3367 [gr-qc].
- [12] S. Weinberg, John Wiley and Sons, New York (1972) pp145.
- [13] Z. Kakushadze, Phys. Rev. D **77** (2008) 024001. arXiv:0710.1061 [hep-th].
- [14] R. Brout, arXiv:0201060 [gr-qc].